

## CHAPTER 2

### 1 SINGLE-PHASE FLUID EQUATIONS

In this chapter, starting from general property balance equations and using the Reynolds transport theorem given in Chapter 1 and the Gauss theorem, the local and volume averaged conservation equation for mass, momentum and energy will be derived for a single-phase flow.

#### 1.1 Local Fluid Equations

##### 1.1.1 General Local Conservation Equation

The general local conservation is obtained by using the general property balance principle applied to a material volume,  $V(\tau)$ , bounded by a material surface,  $A(\tau)$ , in a flow field as illustrated in Fig. 2.1. Obviously the element of volume moves with the local fluid velocity,  $\vec{v}$ . Indicating by:

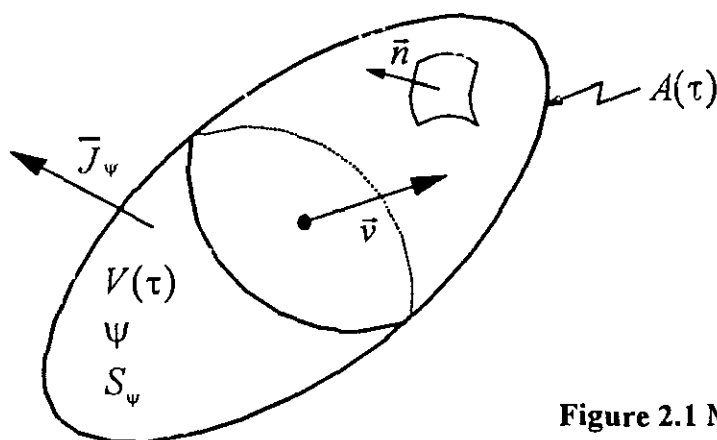


Figure 2.1 Material volume.

- $\Psi$  : property per unit volume of material which may be any scalar, vector or tensor property; for the present application, it will be assigned as mass, momentum, or energy per unit volume,
- $\vec{J}_\psi$  : flow of property per unit area and time across the surface,  $A(\tau)$ , bounding the volume,  $V(\tau)$ ; it may be a vector or a tensor,
- $S_\psi$  : generation of the property per unit volume and time,

the general balance statement for a given property can be written as:

$$\boxed{\text{The time rate change of } \Psi \text{ within } V(\tau)} = \boxed{\text{The net flow of } \Psi \text{ across the } A(\tau) \text{ limiting } V(\tau)} + \boxed{\text{The generation of } \Psi \text{ within } V(\tau)}$$

in other words:

$$\frac{d}{dt} \int_{V(\tau)} \Psi dV = - \int_{A(\tau)} \vec{n} \cdot \vec{J}_{\Psi} dA + \int_{V(\tau)} S_{\Psi} dV \quad (2.1)$$

Using the Reynolds transport theorem given by Eq. 1.33, the above equation can be written as:

$$\int_{V(\tau)} \frac{\partial \Psi}{\partial \tau} dV + \int_{A(\tau)} \vec{n} \cdot \Psi \vec{v} dA = - \int_{A(\tau)} \vec{n} \cdot \vec{J}_{\Psi} dA + \int_{V(\tau)} S_{\Psi} dV. \quad (2.2)$$

Using the Gauss theorem:

$$\int_A \vec{n} \cdot \mathbf{B} dA = \int_V \vec{\nabla} \cdot \mathbf{B} dV \quad (2.3)$$

and by interpreting  $\mathbf{B}$  as  $\Psi \vec{v}$  and  $\vec{J}_{\Psi}$ , the surface integrals appearing in Eq. 2.2 can be transformed into volume integrals:

$$\int_{V(\tau)} \frac{\partial \Psi}{\partial \tau} dV + \int_{V(\tau)} (\vec{\nabla} \cdot \Psi \vec{v}) dV + \int_{V(\tau)} \vec{\nabla} \cdot \vec{J}_{\Psi} dV - \int_{V(\tau)} S_{\Psi} dV = 0 \quad (2.4)$$

Since Eq. 2.4 applies to any volume  $dV$ , it can be deduced that the local instantaneous conservation equation will be in the following form:

$$\frac{\partial \Psi}{\partial \tau} + \vec{\nabla} \cdot \Psi \vec{v} + \vec{\nabla} \cdot \vec{J}_{\Psi} - S_{\Psi} = 0 \quad (2.5)$$

The first term of the above equation is the time rate of change of the property,  $\psi$ , per unit volume, the second term is the rate of convection per unit volume, the third term is the surface flux and the fourth term is the volume source. Table 2.1 gives the most frequently used properties.

### 1.1.2 Fluid Conservation Equations

#### 1) Local continuity equation

The conservation of mass can be expressed in a differential form by setting in Eq. 2.5:

$$\Psi = \rho, \quad \vec{J}_{\Psi} = 0 \quad \text{and} \quad S_{\Psi} = 0$$

This is due to the fact that there is no surface property flux,  $\vec{J}_{\Psi}$  and volume property generation,  $S_{\Psi}$ , with respect to the fixed volume. Therefore, the mass conservation equation has the following form:

Table 2.1  
Most frequently used parameters

Property, $\phi$		Property per unit mass $\psi=\phi/m$	Property per unit volume, $\psi=\phi/V$
Mass	m	1	$\rho$
Volume	V	$1/\rho$	1
Momentum	mv	v	$\rho v$
Kinetic energy	$\frac{1}{2} mv^2$	$\frac{1}{2} v^2$	$\frac{1}{2} \rho v^2$
Potential energy	mgz	gz	$\rho gz$
Internal energy	mu	u	$\rho u$
Total energy	me	e	$\rho e$
Enthalpy	mh	h	$\rho h$
Mass concentration	mc	c	$\rho c$

$$\frac{\partial}{\partial \tau} \rho + \vec{\nabla} \cdot \rho \vec{v} = 0. \quad (2.6)$$

## II) Momentum equation

The conservation equation is obtained from Eq. 2.5 by interpreting:

- the property  $\psi$  as the momentum per unit volume, i.e.,  $\Psi = \rho \vec{v}$ ,
- the flux of the property,  $\vec{J}_\Psi$ , as the surface stress tensor which represents the normal and shear stresses acting on the surface (see Appendix II):

$$\vec{J}_\Psi = -\vec{T} = -(-p \vec{I} + \vec{\sigma}) = p \vec{I} - \vec{\sigma},$$

- the source term,  $S_\Psi$ , as the momentum generated by volumetric forces such as gravity, i.e.:

$$S_\Psi = \rho \vec{g}.$$

The momentum equation has the following form:

$$\frac{\partial}{\partial \tau} \rho \vec{v} + \vec{\nabla} \cdot \rho \vec{v} \vec{v} = -\vec{\nabla} \cdot p \vec{I} + \vec{\nabla} \cdot \vec{\sigma} + \rho \vec{g} \quad (2.7)$$

## III) Conservation of energy

In this case  $\Psi$ ,  $\vec{J}_\Psi$  and  $S_\Psi$  are interpreted as:

- $\Psi$ : total energy which is the sum of the internal and kinetic energies per unit volume, i.e.

$$\Psi = \rho \left( u + \frac{1}{2} \vec{v} \cdot \vec{v} \right),$$

- ♦  $\bar{J}_\Psi$  : heat conduction across the control surface and the work done by surface forces, i.e.

$$\bar{J} = \vec{q}'' - \bar{T} \cdot \vec{v}$$

where

$$\bar{T} = -p \bar{I} + \bar{\sigma}.$$

- ♦  $S_\Psi$  : work done by the volumetric forces and the energy generation per unit volume (due to chemical or nuclear reactions, for example), i.e.

$$S_\Psi = \rho \vec{g} \cdot \vec{v} + \dot{Q}_g$$

Substituting the above interpretations into Eq. 2.5, the total energy equation is obtained as:

$$\begin{aligned} & \frac{\partial}{\partial \tau} \rho \left( u + \frac{1}{2} \vec{v} \cdot \vec{v} \right) + \vec{\nabla} \cdot \rho \left( u + \frac{1}{2} \vec{v} \cdot \vec{v} \right) \vec{v} \\ &= -\vec{\nabla} \cdot \vec{q}'' - \vec{\nabla} \cdot (p \bar{I} \cdot \vec{v}) + \vec{\nabla} \cdot (\bar{\sigma} \cdot \vec{v}) + \rho \vec{g} \cdot \vec{v} + \dot{Q}_g \end{aligned} \quad (2.8)$$

Using the definition of enthalpy:

$$h = u + \frac{p}{\rho} \quad (2.9)$$

the total energy equation becomes:

$$\begin{aligned} & \frac{\partial}{\partial \tau} \rho \left( h + \frac{1}{2} \vec{v} \cdot \vec{v} \right) - \frac{\partial}{\partial \tau} p + \vec{\nabla} \cdot \rho \left( h + \frac{1}{2} \vec{v} \cdot \vec{v} \right) \vec{v} \\ &= -\vec{\nabla} \cdot \vec{q}'' + \vec{\nabla} \cdot (\bar{\sigma} \cdot \vec{v}) + \rho \vec{g} \cdot \vec{v} + \dot{Q}_g \end{aligned} \quad (2.10)$$

The mechanical energy equation is obtained by multiplying scalarly the momentum equation (Eq. 2.7) by the velocity and has the following form:

$$\frac{\partial}{\partial \tau} \left( \frac{1}{2} \rho v^2 \right) + \vec{\nabla} \cdot \frac{1}{2} \rho v^2 \vec{v} = -\vec{v} \cdot \vec{\nabla} p \bar{I} + \vec{v} \cdot (\vec{\nabla} \cdot \bar{\sigma}) + \rho \vec{v} \cdot \vec{g} \quad (2.11)$$

Subtracting the mechanical energy equation from the total energy equation (Eq. 2.8) and taking into account that:

$$\vec{\nabla} \cdot p \vec{v} = p \vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{\nabla} p \quad (2.12)$$

and

$$\vec{\nabla} \cdot (\vec{\sigma} \cdot \vec{v}) = \vec{v} \cdot (\vec{\nabla} \cdot \vec{\sigma}) + \vec{\sigma} : \vec{\nabla} \vec{v} \quad (2.13)$$

the following is obtained for the internal energy equation:

$$\frac{\partial}{\partial \tau}(\rho u) + \vec{\nabla} \cdot \rho u \vec{v} = -\vec{\nabla} \cdot \vec{q}'' - p \vec{\nabla} \cdot \vec{v} + \vec{\sigma} : \vec{\nabla} \vec{v} + \dot{Q}_g \quad (2.14)$$

Subtracting the mechanical energy equation from Eq. 2.10 yields enthalpy equation:

$$\frac{\partial}{\partial \tau}(\rho h) + \vec{\nabla} \cdot \rho h \vec{v} = -\vec{\nabla} \cdot \vec{q}'' + \frac{\partial}{\partial \tau} p + \vec{v} \cdot \vec{\nabla} p + \vec{\sigma} : \vec{\nabla} \vec{v} + \dot{Q}_g \quad (2.15)$$

At this point, the unknown dependent variables are: the specific mass,  $\rho$ , three velocity components  $u$ ,  $v$ ,  $w$ , the pressure,  $p$ , the specific internal energy and the temperature, for a total of seven. The number of available equations are: one continuity (Eq. 2.6), three momentum (Eq. 2.7) and one energy (Eq. 2.8), for a total of five. Thus, the five differential equations must be supplemented by two additional equations which do not introduce any additional unknowns. The additional equations are equations of state given by:

$$\rho = \rho(p, u) \quad (2.16)$$

and

$$T = T(p, u) \quad (2.17)$$

where the independent variables are taken to be the pressure and specific internal energy. Other combinations of dependent and independent variables are also possible. Finally, we note that other properties such as viscosity, conductivity etc., are not constant and they are function of the temperature and pressure.

The development of the local equations given above is not intended to be an exhaustive review of the subject. Further developments to the local conservation equations can be found in fluid mechanics text books.

### ***1.2 Macroscopic Fluid Equations***

The conservation equations (Eqs. 2.6, 2.7 and 2.8) in conjunction with the constitutive heat conduction law (Fourier law) and the state equations (Eqs. 2.16 and 2.17) provide sufficient information to calculate the distribution of the dependent variables throughout the flow field. Additional information must be supplied for the initial conditions of the fluid and the conditions at the boundary of the flow field. However, most engineering analysis are conducted with simplified one dimensional models obtained by area or volume averaging of the local equations. In the process of averaging, information on the detailed distribution of the dependent variables normal to the flow direction is lost; the effect of these distributions are accounted for by the use of empirical constitutive relationships (for example, wall friction and wall heat transfer coefficients for a flow in a pipe). In complex flow geometries such as abrupt expansion and contraction, laterally interconnected parallel subchannels, dividing or converging steams a lot of engineering judgement should

be used during the use of the averaged equations. Continued comparison of the predictions of the area or volume averaged models with experimental data are necessary to determine both the soundness of the approximations associated with the model and the techniques employed in the application of a one-dimensional model to complex multidimensional systems.

### 1.2.1 Generalised Volume Averaged Conservation Equations

The generalised volume averaged conservation equation is obtained by integrating the general local conservation equation (Eq. 2.5) over a geometric volume  $V(\tau)$  and combining it with the generalised transport theorem given in Chapter 1 (Eq. 1.34). The integration of the local conservation equation (Eq. 2.5) over the geometric volume  $V(\tau)$  gives:

$$\int_{V(\tau)} \frac{\partial \Psi}{\partial \tau} dV + \int_{V(\tau)} \vec{\nabla} \cdot \Psi \vec{v} dV + \int_{V(\tau)} \vec{\nabla} \cdot \bar{J}_\Psi dV - \int_{V(\tau)} S_\Psi dV = 0 \quad (2.18)$$

The combination of the above equation with generalised transport theorem (Eq. 1.34) yields:

$$\begin{aligned} \frac{d}{d\tau} \int_{V(\tau)} \Psi dV = & - \int_{V(\tau)} \vec{\nabla} \cdot \Psi \vec{v} dV - \int_{V(\tau)} \vec{\nabla} \cdot \bar{J}_\Psi dV \\ & + \int_{A(\tau)} \Psi \vec{\omega} \cdot \vec{n} dA + \int_{V(\tau)} S_\Psi dV \end{aligned} \quad (2.19)$$

The first two volume integral on the right hand side of the above equation can be transformed into surface integrals by the use of Gauss theorem. Therefore Eq. 2.19 becomes:

$$\frac{d}{d\tau} \int_{V(\tau)} \Psi dV = - \int_{A(\tau)} \vec{n} \cdot \Psi (\vec{v} - \vec{\omega}) dA - \int_{A(\tau)} \vec{n} \cdot \bar{J}_\Psi dA + \int_{V(\tau)} S_\Psi dV \quad (2.20)$$

This is the generalised macroscopic (volume integrated) balance equation. The term on the left hand side of the equation is the time rate change of the property  $\psi$  within the volume  $V(\tau)$ , the first term on the right hand side is the rate of convection throughout the surface  $A(\tau)$  bounding the volume, the second term is the flow of property through the same surface and finally the last term shows the property generated within the volume.

### 1.2.2 Volume Integrated Conservation Equations

#### 1) Macroscopic mass balance

By interpreting:

$$\Psi = \rho, \quad \bar{J}_\Psi = 0 \quad \text{and} \quad S_\Psi = 0$$

the mass conservation equation in the volume  $V(\tau)$  can be expressed as:

$$\frac{d}{d\tau} \int_{V(\tau)} \rho dV = - \int_{A(\tau)} \vec{n} \cdot \rho (\vec{v} - \vec{\omega}) dA \quad (2.21)$$

In the above equation, if  $\vec{\omega} = \vec{v}$  we have a material volume; if  $\vec{\omega} = 0$  we have a fix control volume.

### II) Macroscopic momentum balance

Interpreting:

$$\Psi = \rho \vec{v}, \quad \bar{J}_\Psi = -\bar{T} = p \bar{I} - \bar{\sigma} \quad \text{and} \quad S_\Psi = \rho \vec{g}$$

the macroscopic momentum balance equation is obtained as:

$$\begin{aligned} \frac{d}{d\tau} \int_{V(\tau)} \rho \vec{v} dV = & - \int_{A(\tau)} \vec{n} \cdot \rho \vec{v} (\vec{v} - \vec{\omega}) dA - \int_{A(\tau)} \vec{n} \cdot p \bar{I} dA \\ & + \int_{A(\tau)} \vec{n} \cdot \bar{\sigma} dA + \int_{V(\tau)} \rho \vec{g} dV. \end{aligned} \quad (2.22)$$

### III) Macroscopic total energy balance

Interpreting:

$$\Psi = \rho e = \rho \left( u + \frac{1}{2} \vec{v} \cdot \vec{v} \right), \quad \bar{J}_\Psi = \vec{q}'' - \bar{T} \cdot \vec{v} \quad \text{and} \quad S_\Psi = \rho \vec{g} \cdot \vec{v} + \dot{Q}_g$$

the macroscopic energy balance is obtained as.

$$\begin{aligned} \frac{d}{d\tau} \int_{V(\tau)} \rho e dV = & \int_{A(\tau)} \rho e \vec{n} \cdot (\vec{v} - \vec{\omega}) dA - \int_{A(\tau)} \vec{n} \cdot \vec{q}'' dA \\ & + \int_{A(\tau)} \vec{n} \cdot \bar{T} \cdot \vec{v} dA + \int_{V(\tau)} \rho \vec{g} \cdot \vec{v} dV + \int_{V(\tau)} \dot{Q}_g dV. \end{aligned} \quad (2.23)$$

The volumetric body force,  $\vec{g}$ , which appears in the fourth integral on the right hand side of Eq. 2.23 can be derived from a scalar potential function in the following form:

$$\vec{g} = -\vec{\nabla} \phi. \quad (2.24)$$

Therefore, the work term due to the volumetric force can be written as:

$$\int_{V(\tau)} \rho \vec{g} \cdot \vec{v} dV = - \int_{V(\tau)} \rho \vec{v} \cdot \vec{\nabla} \phi dV \quad (2.25)$$

From vector analysis we know that:

$$\vec{\nabla} \cdot [\phi (\rho \vec{v})] = (\rho \vec{v}) \cdot \vec{\nabla} \phi + \phi \vec{\nabla} \cdot (\rho \vec{v}) \quad (2.26)$$

Substituting Eq. 2.26 into Eq. 2.25 we obtain:

$$\int_{V(\tau)} \rho \vec{g} \cdot \vec{v} dV = -\int_{V(\tau)} \vec{\nabla} \cdot (\phi \rho \vec{v}) dV + \int_{V(\tau)} \phi \vec{\nabla} \cdot \rho \vec{v} dV \quad (2.27)$$

The first term on the right hand side of Eq. 2.27 can be transformed into a surface integral by use of Gauss theorem:

$$\int_{V(\tau)} \vec{\nabla} \cdot (\phi \rho \vec{v}) dV = \int_{A(\tau)} \phi \rho \vec{v} \cdot \vec{n} dA \quad ; \quad (2.28)$$

the second term of the same equation can be rewritten by use of the local continuity equation (Eq. 2.6) as:

$$\int_{V(\tau)} \phi \vec{\nabla} \cdot \rho \vec{v} dV = -\int_{V(\tau)} \phi \frac{\partial \rho}{\partial \tau} dV \quad (2.29)$$

Substituting Eqs. 2.28 and 2.29 into Eq. 2.27 we obtain:

$$\int_{V(\tau)} \rho \vec{g} \cdot \vec{v} dV = -\int_{A(\tau)} \phi \rho \vec{v} \cdot \vec{n} dA - \int_{V(\tau)} \phi \frac{\partial \rho}{\partial \tau} dV \quad (2.30)$$

Using the generalised transport theorem, interpreting  $\Psi$  as  $\rho\phi$  and assuming that the scalar potential  $\phi$  is independent of time, the last term on the right hand side of the above equation can be written as:

$$\int_{V(\tau)} \phi \frac{\partial \rho}{\partial \tau} dV = \frac{d}{d\tau} \int_{V(\tau)} \rho \phi dV - \int_{A(\tau)} \rho \phi \vec{\omega} \cdot \vec{n} dA \quad (2.31)$$

Combining Eqs. 2.30 and 2.31, we obtain:

$$\int_{V(\tau)} \rho \vec{g} \cdot \vec{v} dV = -\int_{A(\tau)} \rho \phi (\vec{v} - \vec{\omega}) \cdot \vec{n} dA - \frac{d}{d\tau} \int_{V(\tau)} \rho \phi dV \quad (2.32)$$

Substituting Eq. 2.32 into Eq. 2.23 we get:

$$\begin{aligned} \frac{d}{d\tau} \int_{V(\tau)} \rho(e + \phi) dV &= -\int_{A(\tau)} \rho(e + \phi) \vec{n} \cdot (\vec{v} - \vec{\omega}) dA \\ &\quad - \int_{A(\tau)} \vec{n} \cdot \vec{q}'' dA + \int_{A(\tau)} \vec{n} \cdot \vec{T} \cdot \vec{v} dA + \int_{V(\tau)} \dot{Q}_s dV \end{aligned} \quad (2.33)$$

where the heat flux is given by the Fourier law of conduction:

$$\vec{q}'' = -k \vec{\nabla} t \quad (2.34)$$

In order to close the above conservation equations, two state equations as given by Eqs. 2.16 and



2.17 must also be supplied. Finally, the fluid transport properties such as viscosity and conductivity are also needed.